Equilibrium and Infeasibility: A new solution concept for games

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***Abstract*— Addressing infeasibility in non-cooperative games has become an important topic, as many problems across different applications face this issue. We propose a new solution concept for generalized games with possibly infeasible individual constraints. A solution is defined as the limit of a sequence of generalized Nash equilibria induced by games with penalty terms relaxing the individual constraints. Existence is established for a broad range of games and we provide conditions allowing to characterize a ψ-penalized solution as a strategy profile maximizing every player's utility over all her penalty minimizing strategies. A variation of Divide-the-Dollar serves as an illustrative example.**

I. INTRODUCTION

Originally introduced to model a competitive economy [12], [2], generalized games have also been used to study various applications (e.g., [15], [20], [5]). These games are usually solved by using the *generalized Nash equilibrium* (GNE) [12] defined as a strategy profile that is feasible for all players and without any feasible benefiting deviation. We refer to [16] for an extensive survey. All standard existence theorems [14] assume that the constraint correspondences are nonempty-valued. Under this assumption, a player has always at least one feasible strategy. Moreover, it is reasonable to assume that she prefers feasibility implying that infeasible profiles are not stable and that the GNE is an appropriate solution concept. Typically, constraint correspondences take only nonempty values in case of a *coupled constraint* [23], i.e. the constraint correspondences are induced by a common set for all players, but this may not hold for general *individual constraints*. Nevertheless, [6] stated the importance of providing solutions to games with constraint correspondences taking also empty values, particularly those without a GNE. To address this problem, [6] considers the set of jointly feasible strategy profiles induced by the individual constraints and solves the corresponding game with coupled constraint. However, this approach relies on the assumption that there is a jointly feasible strategy profile and includes only feasible strategy profiles as solutions. It remains thus an open question if strategy profiles that are infeasible for some players could also be considered credible.

The algorithm proposed in [17], based on a penalization of the constraints, computes an accumulation point that is a GNE or a possibly infeasible generalized stationary point. It can also be used for games with constraint correspondences with empty values or no jointly feasible strategy profile. Nevertheless, it was only applied to games with a GNE and the possibility and consequences of defining all accumulations points as a new solution concept has not been discussed.

Essentially, each player is assumed to maximize her utility under her individual constraints and if the set of feasible strategies is empty, she faces an infeasible optimization problem. Solving such problems has also received little attention. An approach [9], consists in detecting infeasibility and proposing a feasible model. Another approach is to design algorithms converging to a reasonable point even for infeasible problems (e.g., [7], [8], [11], [1]).

In this paper, we introduce a new solution concept for generalized games, including those with constraint correspondences taking empty values, without a jointly feasible strategy profile or without a GNE. We relax the individual constraints and incorporate them into the utility function by a penalty function ψ [10]. Iteratively increasing the penalty induces a sequence of *ψ-penalized games* and corresponding GNEs, a *ψ-penalized solution* being defined as their limit. The formal definition is given in Section II and our main results in Section III. Finally, we show an illustrative example in Section IV. All notations used in this article can be found in Table I.

II. MODEL AND CONCEPT

We consider a N-player non-cooperative generalized game $G := (\mathcal{N}, (\mathcal{A}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}}, (C_i)_{i\in\mathcal{N}})$. The set of players is described by the set $\mathcal{N}$ and the set of strategies available to each player $i \in \mathcal{N}$ by $\mathcal{A}_i$. Following standard notation, we write $\mathcal{A}$ for the product $\prod_{j\in\mathcal{N}} \mathcal{A}_j$ of all strategy sets and $\mathcal{A}_{-i}$ for the product $\prod_{j\in\mathcal{N}, j\neq i} \mathcal{A}_j$ of all strategy sets except the one of player i. The set of feasible strategies for each player $i \in \mathcal{N}$ is given by a *individual constraint correspondence* $C_i : \mathcal{A}_{-i} \to \mathcal{P}(\mathcal{A}_i)$ which associates to each strategy profile $a_{-i} \in \mathcal{A}_{-i}$ of the other players the feasible strategies $C_i(a_{-i})$ for player i. We emphasize that the set of feasible strategies is thus dependent on the choice of strategies of the other players. Throughout this article, we assume that for all players $i \in \mathcal{N}$ the strategy set $\mathcal{A}_i$ is a convex and compact subset of $\mathbb{R}$. Moreover, we assume that for all players the utility function u_i is continuous and concave in a_i, i.e., for all $a_{-i} \in \mathcal{A}_{-i}$, $u_i(\cdot, a_{-i})$ is concave. Additionally, we consider a convex and compact *coupled constraint* $\mathcal{L} \subseteq A$ that is shared by all player and induces for each player $i \in \mathcal{N}$ the *coupled constraint correspondence* $L_i : \mathcal{A}_{-i} \to \mathcal{P}(\mathcal{A}_i)$ such that $L_i(a_{-i}) := \{a_i \in \mathcal{A}_i \mid (a_i, a_{-i}) \in \mathcal{L}\}$. Without loss of generality, we assume that $\bigcup_{a_{-i}\in\mathcal{A}_{-i}} L_i(a_{-i}) = \mathcal{A}_i$ (see [23]) and that for all players $i \in \mathcal{N}$ and strategy profiles $a_{-i} \in \mathcal{A}_{-i}$, $C_i(a_{-i}) \subseteq L_i(a_{-i})$.

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The aim of each player $i \in \mathcal{N}$ is to maximize her utility function $u_i : \mathcal{A} \to \mathbb{R}$ while respecting her constraints, i.e. to chose a strategy $a_i \in \mathcal{A}_i$, given the other players' strategies a_{-i}, that solves the maximization problem

$$\begin{aligned} &\max_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i}) \\ &\text{s.t. } a_i \in C_i(a_{-i}) \end{aligned} \tag{1}$$

We emphasize that we do not assume that the individual constraints are nonempty-valued and Problem (1) may be infeasible for some strategy profile $a_{-i} \in \mathcal{A}_{-i}$. We propose therefore to relax the individual constraints and incorporate a penalty function into the utility.

A. ψ-penalized game

We introduce the N-player generalized *ψ-penalized game* $G_\rho := (\mathcal{N}, (\mathcal{A}_i)_{i\in\mathcal{N}}, (u_i - \rho\psi_i)_{i\in\mathcal{N}}, (L_i)_{i\in\mathcal{N}})$, an auxiliary parametrized game with coupled constraints and penalized utilities induced by game G. As before, the set of players is $\mathcal{N}$, the set of strategies of player $i \in \mathcal{N}$ is $\mathcal{A}_i$ but the only constraints are given by the coupled constraint correspondences L_i. The individual constraints are relaxed and replaced by a penalty term that consists of a penalty parameter $\rho > 0$ and a penalty function $\psi_i : \mathcal{A} \to \mathbb{R}_{\geq 0}$ such that for all $i \in \mathcal{N}$ and $a \in \mathcal{A}$

$$a_i \in \overline{C_i(a_{-i})} \Leftrightarrow \psi_i(a) = 0 \tag{2}$$

Furthermore, we assume that ψ_i is continuous and convex in a_i, i.e., for all $a_{-i} \in \mathcal{A}_{-i}$, $\psi_i(\cdot, a_{-i})$ is convex. The utility function of player i in G_ρ is defined as the sum of the original utility and a penalty term, i.e. $u_i - \rho\psi_i$, such that, for any $a_{-i} \in \mathcal{A}_{-i}$, player i solves the maximization problem

$$\begin{aligned} &\max_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i}) - \rho\psi_i(a_i, a_{-i}) \\ &\text{s.t. } a_i \in L_i(a_{-i}) \end{aligned}$$

We only relaxed the individual constraints C_i, while the coupled constraint $\mathcal{L}$ is left unchanged. As discussed earlier, games with only a coupled constraint $\mathcal{L}$ have been extensively studied [23], [14] and do not bear the difficulty of infeasibility. We follow hereby the idea of a partial penalization presented in [18].

B. ψ-penalized solution concept

Let $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ be any sequence such that $\rho_n \to \infty$ and $(G_{\rho_n})_{n\in\mathbb{N}}$ the corresponding sequence of ψ-penalized games (as defined in Section II-A). In the following, we show that our assumptions imply that each game G_{ρ_n} has at least one GNE $a_n^* \in \mathcal{A}$ (see Theorem 1) and define a *ψ-penalized solution* a^* of game G as an accumulation point (see Definition 1.1, p. 209 in [13]) of any such sequence $(a_n^*)_{n\in\mathbb{N}}$.

We now give the formal definition of a ψ-penalized solution of game G.

Definition 1: A strategy profile a^* is called a *ψ-penalized solution* of game G *under the constraint* $\mathcal{L}$ if there is a sequence $(a_n^*)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, a_n^* is a GNE of the ψ-penalized game G_{ρ_n} and a^* is an accumulation point of $(a_n^*)_{n\in\mathbb{N}}$.

A ψ-penalized solution may be infeasible for certain players and does not, in contrast to the GNE, demand feasibility for all players. It provides thus a solution to generalized games with constraint correspondences taking empty values or defining disjoint feasible sets. We show that a ψ-penalized solution exists under the initially stated assumptions.

Theorem 1: There exists a ψ-penalized solution of game G.

Proof: For all players $i \in \mathcal{N}$ and any $\rho \in \mathbb{R}_{>0}$, the utility function $u_i - \rho\psi_i$ of the ψ-penalized game G_ρ is continuous as the composition of two continuous functions u_i and ψ_i. Moreover, for all $a_{-i} \in \mathcal{A}_{-i}$, $u_i(\cdot, a_{-i}) - \rho\psi_i(\cdot, a_{-i})$ is concave as the sum of the two concave functions $u_i(\cdot, a_{-i})$ and $-\rho\psi_i(\cdot, a_{-i})$. Hence, for all $n \in \mathbb{N}$, the ψ-penalized game G_{ρ_n} is a concave game under the coupled convex and compact constraint $\mathcal{L}$ and has a GNE a_n^* (see Theorem 1 in [23]). We can thus construct a sequence $(a_n^*)_{n\in\mathbb{N}}$ such that a_n^* is a GNE of ψ -penalized game G_{ρ_n}. Since $\mathcal{L}$ is compact, we know that $(a_n^*)_{n\in\mathbb{N}}$ has an accumulation point (Bolzano–Weierstrass theorem, see 3.4.8 in [3]). ■

We remark that, by definition, the ψ-penalized solution depends on $\psi := (\psi_i)_{i\in\mathcal{N}}$ and the associated sequence of ψ-penalized games $(G_{\rho_n})_{n\in\mathbb{N}}$. The dependence on exogenous parameters can be found in other solution concepts, as the *normalized equilibrium* on the chosen Lagrange multipliers [23] or the *H-essential equilibrium* on the class of penalty functions [25]. The properties of ψ-penalized solutions with respect to ψ, beyond the results shown in this paper, remain an open question (see Section V).

III. Main Results

In this section, we show that a ψ-penalized solution can be characterized without relying on the sequence of ψ-penalized games $(G_{\rho_n})_{n\in\mathbb{N}}$. First, in Section III-A, we provide conditions implying that a ψ-penalized solution $a^* \in \mathcal{A}$ solves for each player $i \in \mathcal{N}$

$$\begin{aligned} &\max_{a_i} u_i(a_i, a_{-i}^*) \\ &\text{s.t. } a_i \in \underset{b_i \in L_i(a_{-i}^*)}{\arg\min}\ \psi_i(b_i, a_{-i}^*) \end{aligned} \tag{3}$$

Thus, at a ψ-penalized solution a benefiting deviation in u_i would result in an infeasible strategy with respect to the coupled constraint $\mathcal{L}$ or a higher penalty ψ_i.

In Section III-B, we give a counterexample showing that in general it is not sufficient for a strategy profile to solve Problem (3) for each player to be a ψ-penalized solution. Finally, in Section III-C, we provide sufficient (but not necessary) conditions.

In the following, we have used the notion of *lower semi-continuous* (l.s.c.) defined in [4]. We recall the definition in Appendix V, where we also present a preliminary result for the following proof.

A. ψ-penalized solutions solving Problem (3)

First, we show that a ψ-penalized solution of game G minimizes the penalty function ψ for every player in response to the strategies of the other players.

Lemma 2: Let $a^* \in \mathcal{A}$ be a ψ-penalized solution of game G. If for some player $i \in \mathcal{N}$ the coupled constraint correspondence L_i is l.s.c., then $a_i^* \in \arg\min_{L_i(a_{-i}^*)} \psi_i(\cdot, a_{-i}^*)$.

Proof: By contradiction, we assume there is a strategy $a_i \in \mathcal{A}_i$ such that $a_i \in \mathcal{L}_i(a_{-i}^*)$ and

$$\psi_i(a_i^*, a_{-i}^*) - \psi_i(a_i, a_{-i}^*) > 0 \tag{4}$$

a^* is a ψ-penalized solution of game G, thus by definition there is a sequence $(a_{n_k}^*)_{k\in\mathbb{N}} \subset \mathcal{A}$ such that, for all $k \in \mathbb{N}$, $a_{n_k}^*$ is a GNE of $G_{n_k} := G_{\rho_{n_k}}$ and converging to a^*.
Moreover, since L_i is l.s.c., and thus l.s.c. at a_{-i}^*, there is a sequence $(a_{n_k,i})_{k\in\mathbb{N}}$ such that $a_{n_k,i} \in L_i(a_{n_k,-i}^*)$ and $a_{n_k,i} \to a_i$ as $k \to \infty$ (Lemma 6). By our initial assumption (4) and the continuity of ψ_i, we deduce

$$\begin{aligned}
0 &< \lim_{k\to\infty} \psi_i(a_{n_k,i}^*, a_{n_k,-i}^*) - \psi_i(a_{n_k,i}, a_{n_k,-i}^*) \\
&= \lim_{k\to\infty} \frac{1}{\rho_{n_k}} \Big[\big(u_i(a_{n_k,i}, a_{n_k,-i}^*) - \rho_{n_k}\psi_i(a_{n_k,i}, a_{n_k,-i}^*)\big) \\
&\quad - \big(u_i(a_{n_k,i}^*, a_{n_k,-i}^*) - \rho_{n_k}\psi_i(a_{n_k,i}^*, a_{n_k,-i}^*)\big)\Big] \\
&\quad + \frac{1}{\rho_{n_k}} \Big[u_i(a_{n_k,i}^*, a_{n_k,-i}^*) - u_i(a_{n_k,i}, a_{n_k,-i}^*)\Big]
\end{aligned}$$

Moreover, since $a_{n_k}^*$ is a GNE of G_{n_k} for all $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\big(u_i(a_{n_k,i}, a_{n_k,-i}^*) - \rho_{n_k}\psi_i(a_{n_k,i}, a_{n,-i}^*)\big) \\
&\quad - \big(u_i(a_{n_k,i}^*, a_{n_k,-i}^*) - \rho_{n_k}\psi_i(a_{n_k,i}^*, a_{n_k,-i}^*)\big) \le 0
\end{aligned}$$

Furthermore, u_i is continuous and $\mathcal{L}$ is compact, thus the difference $u_i(a_{n_k,i}^*, a_{n_k,-i}^*) - u_i(a_{n_k,i}, a_{n_k,-i}^*)$ is bounded by some constant $C > 0$, i.e. for all $k \in \mathbb{N}$:

$$|u_i(a_{n_k,i}^*, a_{n_k,-i}^*) - u_i(a_{n_k,i}, a_{n_k,-i}^*)| \le C$$

We conclude

$$\begin{aligned}
0 &< \lim_{k\to\infty} \frac{1}{\rho_{n_k}} \Big[\big(u_i(a_{n_k,i}, a_{n_k,-i}^*) - \rho_{n_k}\psi_i(a_{n_k,i}, a_{n_k,-i}^*)\big) \\
&\qquad - \big(u_i(a_{n_k,i}^*, a_{n_k,-i}^*) - \rho_{n_k}\psi_i(a_{n_k,i}^*, a_{n_k,-i}^*)\big)\Big] \\
&\qquad + \frac{1}{\rho_{n_k}} \Big[u_i(a_{n_k,i}^*, a_{n_k,-i}^*) - u_i(a_{n_k,i}, a_{n_k,-i}^*)\Big] \\
&\le 0 + \lim_{k\to\infty} \frac{1}{\rho_{n_k}} \cdot C = 0
\end{aligned}$$

This is a contradiction. ■

Now, we show that under additional conditions on the regularity of the individual constraints, a ψ-penalized solution of game G maximizes the original utility function among all minimizers of the penalty function.

Theorem 3: Assume that for player $i \in \mathcal{N}$ the following conditions hold

(i) L_i is l.s.c.,
(ii) for all $a_{-i} \in \mathcal{A}_{-i}$, C_i is l.s.c. in a_{-i} or $C_i(a_{-i})$ is a singleton,
(iii) for all $a_{-i} \in \mathcal{A}_{-i}$, $\psi_i(\cdot, a_{-i})$ is strictly convex on $L_i(a_{-i}) \setminus C_i(a_{-i})$.

If a^* is ψ-penalized solution of game G then a_i^* solves Problem (3) for player i.

Proof: We distinguish two cases, $C_i(a_{-i}^*) = \emptyset$ (i.e. player i has no feasible strategy at a_{-i}^*) and $C_i(a_{-i}^*) \neq \emptyset$ (i.e. player i has a feasible strategy at a_{-i}^*).
First Case. Let $C_i(a_{-i}^*) = \emptyset$. By Assumption (iii), $\psi_i(\cdot, a_{-i}^*)$ is strictly convex on $L_i(a_{-i}^*)$ implying that $\arg\min_{L_i(a_{-i}^*)} \psi_i(\cdot, a_{-i}^*)$ is a singleton. By Lemma 2, a_i^* is this unique solution implying that it also solves Problem (3) as unique element in the feasible set.
Second Case. Let $C_i(a_{-i}^*) \neq \emptyset$. By construction of ψ_i (see (2)) and Lemma 2, $a_i^* \in \arg\min_{L_i(a_{-i}^*)} \psi_i(\cdot, a_{-i}^*) = \overline{C_i(a_{-i}^*)}$, thus we must show that for all $b_i \in \overline{C_i(a_{-i}^*)}$

$$u_i(a_i^*, a_{-i}^*) \ge u_i(b_i, a_{-i}^*)$$

We consider the two cases induced by Assumption (ii).
If $C_i(a_{-i}^*)$ is a singleton, then $\overline{C_i(a_{-i}^*)} = C_i(a_{-i}^*)$ and, as before, a_i^* is the unique solution to Problem (3).
If C_i is l.s.c. at a_{-i}^* then the correspondence $\overline{C} : \mathcal{A}_{-i} \to \mathcal{P}(\mathcal{A}_i)$ s.t. for all $a_{-i} \in \mathcal{A}_{-i}$ $\overline{C}(a_{-i}) := \overline{C_i(a_{-i})}$ is also l.s.c. (see Lemma 7 in Appendix V).
Besides, by definition of the ψ-penalized solution, there is a sequence $(a_{n_k}^*)_{k\in\mathbb{N}}$ converging to a^* as $k \to \infty$ such that, for all $k \in \mathbb{N}$, $a_{n_k}^*$ is a GNE of $G_{n_k} := G_{\rho_{n_k}}$. Furthermore, by Lemma 6, for all $b_i \in \overline{C_i}(a_{-i}^*) = \arg\min_{L_i(a_{-i}^*)} \psi_i(\cdot, a_{-i}^*)$ there is a sequence $(b_{n_k,i})_{n\in\mathbb{N}}$ such that $b_{n_k,i} \in \overline{C_i}(a_{n_k,-i}^*)$ and $b_{n_k,i} \to b_i$ as $k \to \infty$. We have

$$\begin{aligned}
&u_i\big(b_{n_k,i}, a_{n_k,-i}^*\big) \\
&\quad = u_i\big(b_{n_k,i}, a_{n_k,-i}^*\big) - \rho_{n_k}\psi_i\big(b_{n_k,i}, a_{n_k,-i}^*\big) \\
&\quad \le u_i\big(a_{n_k,i}^*, a_{n_k,-i}^*\big) - \rho_{n_k}\psi_i\big(a_{n_k,i}^*, a_{n_k,-i}^*\big) \\
&\quad \le u_i\big(a_{n_k,i}^*, a_{n_k,-i}^*\big)
\end{aligned}$$

The first equality follows from the penalty vanishing at $(b_{n_k,i}, a_{n_k,-i}^*)$ as $b_{n_k,i} \in \overline{C_i}(a_{n_k,-i}^*)$. The second line holds since $a_{n_k}^*$ is a GNE of G_{n_k} and $b_{n_k,i} \in \overline{C_i}(a_{-i,n}^*) \subset L_i(a_{-i,n}^*)$. The third one follows by nonnegativity of ψ_i and ρ_n. Finally, u_i is continuous and the sequences $(a_{n_k}^*)_{k\in\mathbb{N}}$ and $(b_{n_k,i})_{n\in\mathbb{N}}$ are convergent, thus by passing to the limit we have $u_i\big(b_i, a_{-i}^*\big) \le u_i\big(a_i^*, a_{-i}^*\big)$, implying that a_i^* solves Problem (3). ■

Remark 1: It is standard in the literature to assume that the constraint correspondences are l.s.c. and nonempty valued, i.e. for all $i \in \mathcal{N}$ and for all $a_{-i} \in \mathcal{A}_{-i}$, $C_i(a_{-i}) \neq \emptyset$. While in Theorem 3, we also assume lower semi-continuity by Assumption (i), Assumption (ii) does not imply that the individual correspondences take nonempty values. In fact, by definition, a correspondence is l.s.c. in points with empty values but it is not at a transition point from empty to nonempty values, i.e. for any topological spaces X, Y a correspondence $C : X \to Y$ is not l.s.c. in a point $x_0 \in X$ such that $C(x_0) \neq \emptyset$ but for all neighborhoods $U(x_0)$ of x_0 there is a $x \in U(x_0)$ such that $C(x) = \emptyset$. To address this problem, by Assumption (ii), we demand that the correspondence associates a singleton to such transition points which guarantees, roughly spoken, a "smooth" transition from empty to nonempty values. Moreover, we remark that Assumption (ii) was imposed to guarantee the uniqueness of the minimizer of $\psi_i(\cdot, a_{-i})$ on $L_i(a_{-i}) \setminus C_i(a_{-i})$ and could therefore be replaced by assuming directly the uniqueness.

Furthermore, the following corollary follows by the second case in the proof of Theorem 3 showing that every ψ-penalized solution of game G that is feasible for all players is a GNE.

Corollary 4: Let a^* be a ψ-penalized solution of game G. If for all players $i \in \mathcal{N}$, $a_i^* \in C_i(a_{-i}^*)$, L_i is l.s.c. and C_i is l.s.c. or a singleton at a_{-i}^*, then a^* is a GNE of game G.

B. Solutions of Problem (3) *may not be ψ-penalized solutions: a counterexample*

In this section, we provide a counterexample showing that, given penalty function ψ, it is not sufficient for a point to solve Problem (3) for every player to be a ψ-penalized solution. Therefore, we consider a 2-player game G^{ce} with strategy sets $\mathcal{A}_i = [0, 1]$ and utilities $u_i(a_i, a_{-i}) = -(1 - a_i)^2$. The coupled constraint is given by $\mathcal{L} = \{(a_1, a_2) \in [0, 1]^2 \mid a_1 + a_2 \leq 1\}$ and the individual constraints of the players by

$$C_1(a_2) = \begin{cases} \{a_2\} & a_2 \in [0, 0.5] \\ \emptyset & a_2 \in (0.5, 1] \end{cases}$$

and

$$C_2(a_1) = \begin{cases} [a_1, \min(3a_1, 1 - a_1)] & a_1 \in [0, 0.5] \\ \emptyset & a_1 \in (0.5, 1] \end{cases}$$

We consider the penalty functions $\psi_1(a_1, a_2) = (a_1 - a_2)^2$ and $\psi_2(a_1, a_2) = d((a_1, a_2), \Gamma(C_2))^2$ where d is the distance to a set defined in Table I. The set of GNEs of the ψ-penalized game $G^{\text{ce}}_{\rho_n}$ is

$$\left\{(a_1, 1 - a_1) \in [0, 1]^2 \mid \frac{\rho_n}{4\rho_n + 10} \leq a_1 \leq \frac{\rho_n + 1}{2\rho_n + 1}\right\}$$

Thus, the set of ψ-penalized solutions of game G^{ce} (see Fig. 1) is

$$\left\{(a_1, 1 - a_1) \in [0, 1]^2 \mid \frac{1}{4} \leq a_1 \leq \frac{1}{2}\right\}$$

The point $(0, 0)$ solves Problem (3) for both players but is not a ψ-penalized solution of G^{ce} implying that solving Problem (3) for all players is not sufficient without further assumptions. Moreover, $(0, 0)$ is a GNE of G^{ce}, showing that a GNE of the original game may not be a ψ-penalized solution.

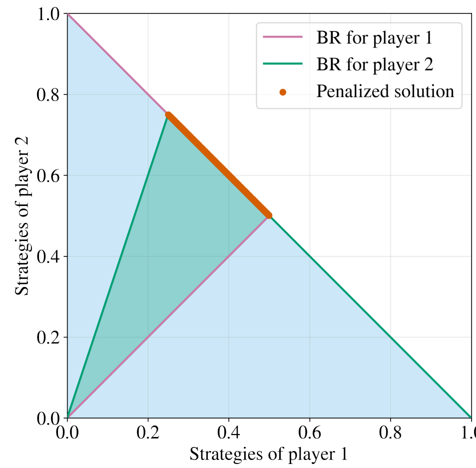


Fig. 1. ψ-penalized solutions (red) and best response correspondences for player 1 (purple) and player 2 (green) under the coupled constraint $\mathcal{L}$ (blue) and their respective individual constraints for game G^{ce}.

C. Sufficient conditions and exact penalization

In this section, we give conditions under which every strategy profile solving Problem (3) for all players is a ψ-penalized solution.

Theorem 5: Let $a^* \in L$. Assume there are constants $\alpha_u, \alpha_\psi > 0$ such that for all players $i \in \mathcal{N}$ and strategies $a_i \in L_i(a_{-i}^*)$, $u_i(\cdot, a_{-i}^*)$ is α_u-Lipschitz continuous at a_i^*, i.e.

$$|u_i(a_i, a_{-i}^*) - u_i(a^*)| \leq \alpha_u |a_i - a_i^*| \tag{5}$$

and

$$\begin{aligned} |\psi_i(a_i, a_{-i}^*) - \psi_i(a^*)| &\geq \alpha_\psi |a_i - a_i^*| \\ \text{or } \psi(a^*) &= \psi(a_i, a_{-i}^*) \end{aligned} \tag{6}$$

If a^* solves Problem (3) for all players, then a^* is a ψ-penalized solution.

Proof: Let a^* be a solution to Problem 3 for all players. We show that a^* is a GNE of every ψ-penalized game G_{ρ_n} with $\rho_n > \alpha_u / \alpha_\psi$. Let $i \in \mathcal{N}$ and $a_i \in L_i(a_{-i}^*)$. By definition of a^* we have $\psi_i(a_i, a_{-i}^*) - \psi_i(a^*) \geq 0$.
First case. Let $\psi_i(a_i, a_{-i}^*) = \psi_i(a^*)$. Then, again by definition of a^*, we also have $u_i(a^*) \geq u_i(a_i, a_{-i}^*)$. Implying,

$$u_i(a^*) - \rho_n \psi_i(a^*) \geq u_i(a_i, a_{-i}^*) - \rho_n \psi_i(a_i, a_{-i}^*)$$

Second case. Let $\psi_i(a_i, a_{-i}^*) > \psi_i(a^*)$ and $u_i(a^*) \geq u_i(a_i, a_{-i}^*)$. For all $\rho_n > 0$ we have

$$u_i(a^*) - \rho_n \psi_i(a^*) > u_i(a_i, a_{-i}^*) - \rho_n \psi_i(a_i, a_{-i}^*)$$

Third case. Let $\psi_i(a_i, a_{-i}^*) > \psi_i(a^*)$ and $u_i(a^*) < u_i(a_i, a_{-i}^*)$. For all $\rho_n > \alpha_u / \alpha_\psi$ we have

$$\begin{aligned} \rho_n \Big(\psi_i(a_i, a_{-i}^*) - \psi_i(a^*)\Big) &> \frac{\alpha_u}{\alpha_\psi} \Big(\psi_i(a_i, a_{-i}^*) - \psi_i(a^*)\Big) \\ &\geq \alpha_u |a_i - a_i^*| \geq |u_i(a_i, a_{-i}^*) - u_i(a^*)| \\ &= u_i(a_i, a_{-i}^*) - u_i(a^*) \end{aligned}$$

where the second and third inequalities follow from (6) and (5). Thus, we conclude that a^* is a GNE for all ψ-penalized games G_{ρ_n} with $\rho_n > \alpha_u/\alpha_\psi$. We can thus construct a sequence $(a_n)_{n\in\mathbb{N}}$ converging to a^* such that a_n is a GNE of G_{ρ_n} by setting $a_n = a^*$ for all n such that $\rho_n > \alpha_u/\alpha_\psi$. ■

Remark 2: Condition (6) guarantees that the penalty is either constant or grows at least linearly. In the counterexample in III-B, $\psi_1(\cdot, 0)$ does not satisfy Condition (6) at 0. In fact, $\psi_1(\cdot, 0)$ is differentiable at 0 which minimizes $\psi_1(\cdot, 0)$ implying $\psi_1'(0, 0) = 0$ and preventing the existence of $\alpha_\psi > 0$.

Moreover, we can show similarly that if for some $a^* \in \mathcal{L}$ the conditions hold then, for any pair $\rho_n, \rho_m > \alpha_u/\alpha_\psi$, a^* is a GNE of G_{ρ_n} if and only if a^* is a GNE of G_{ρ_m}. If the conditions are satisfied at all strategy profiles $a \in \mathcal{A}$, then for all $\rho_n, \rho_m > \alpha_u/\alpha_\psi$ the set of GNEs is the same for ψ-penalized games G_{ρ_n} and G_{ρ_m} and the set of ψ-penalized solutions is the closure of the set of GNEs of G_{ρ_n} for $\rho_n > \alpha_u/\alpha_\psi$.

IV. Example

We show now an illustrative example inspired by Divide-the-Dollar that allows an exact penalization. Therefore, we consider a 2-player game G^{DD} in which each player $i \in \{1, 2\}$ makes a demand $\mathcal{A}_i = [0, 1]$. Their utility $u_i : \mathcal{A} \to \mathbb{R}$ is given by their respective demand, i.e. $u_i(a_i, a_{-i}) = a_i$. Both players must share a resource whose limited amount is modeled as a coupled constraint

$$\mathcal{L} = \left\{(a_1, a_2) \in [0, 1]^2 \mid a_1 + a_2 \leq 1\right\}$$

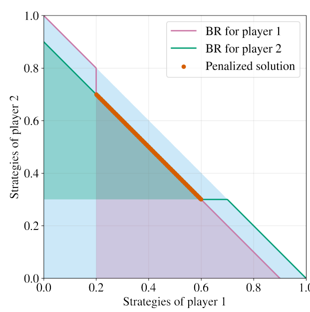
We assume that both players know the amount of the resource and, following the standard way to consider coupled constraints in generalized games, that player i's feasible set of strategies is given by the constraint correspondences L_i. Besides, we assume that each player has additional personal constraints. First, each player i requires at least $\delta_i > 0$. Second, a player means to leave unused an amount $\epsilon_i > 0$ of the resource. The individual constraint correspondence $C_i : [0, 1] \to [0, 1]$ is thus given by $C_i(a_{-i}) := [\delta_i, 1 - \epsilon_i - a_{-i}]$. We consider the penalty function $\psi_i : [0, 1]^2 \to [0, \infty)$ defined by

$$\psi_i(a) = d\big(a_i, \Gamma(C_i)\big) := \inf_{b_i \in \Gamma(C_i)} |a_i - b_i| \tag{7}$$

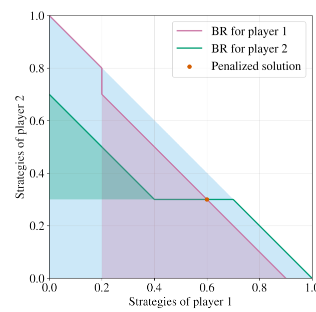
In this example, we consider the penalty function (7) as a natural metric to measure the distance of a point to a non-empty compact set (see the smoothing in [22]). We remark that it is sufficient to consider the GNEs of G_ρ with $\rho > 1/\sqrt{2}$, each point satisfying the conditions of Theorem 5 with $\alpha_u = 1$ and $\alpha_\psi = \sqrt{2}$.

In the following, we will distinguish two cases. First, we assume that both players intend to leave unused the same amount of the resource, i.e. $\epsilon_1 = \epsilon_2$. The individual constraints have a nonempty intersection in which all profiles are feasible for both players (see Fig. 2a). Hence, the set of ψ-penalized solutions is given by the common upper right boundary of the individual constraints and coincides with the set of GNEs of game G^{DD} (see Fig. 2a).

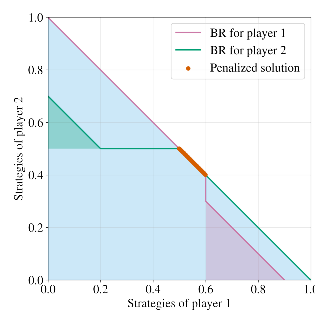
Second, we assume the players differ in how much they leave unused, i.e. $\epsilon_1 \neq \epsilon_2$. Without loss of generality, let $\epsilon_1 < \epsilon_2$. If the δ_1 and δ_2 are small enough, i.e. $\delta_1 + \delta_2 \leq 1 - \epsilon_1$, there is again a nonempty intersection of the individual constraints (see Fig. 2b). All profiles in this intersection are feasible for both players. However, no profile in this intersection is maximizing the utility for player 1 who can always benefit by deviating to a higher feasible demand. As a consequence, game G^{DD} has no GNE. Nevertheless, the game has the ψ-penalized solution $(0.6, 0.3)$. If the requirements δ_1 and δ_2 are large, i.e. $\delta_1 + \delta_2 > 1 - \epsilon_1$, the individual constraints are disjoint (see Fig. 2c) and, by definition, there is no GNE of game G^{DD}. There is an infinity of ψ-penalized solutions on the upper right boundary of the coupled constraint which are all infeasible for both players.



(a) $\delta_1 = 0.2$, $\delta_2 = 0.3$, $\epsilon_1 = \epsilon_2 = 0.1$



(b) $\delta_1 = 0.2$, $\delta_2 = 0.3$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.3$



(c) $\delta_1 = 0.6$, $\delta_2 = 0.5$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.3$

Fig. 2. ψ-penalized solutions (red) and best response correspondences for player 1 (purple) and player 2 (green) under the coupled constraint $\mathcal{L}$ (blue) and their respective individual constraints for game G^{DD}.

V. Future Works

In the future, we aim to generalize our main results and find weaker conditions under which solutions to Problem (3) are ψ-penalized solutions, and vice versa. In particular, it remains an open question to determine conditions such that strategy profiles solving Problem (3) for all players are ψ-penalized solutions without implying an exact penalization, or allowing also differentiable penalty functions (as in Section III-C). Furthermore, characterizing the set of ψ-penalized solutions (e.g., topological structure, robustness, fairness) with respect to penalty function ψ is an important area for further research. Besides, further works should study the conditions for uniqueness of the ψ-penalized solution as well as refinements. Finally, the compatibility of the ψ-penalized solution with established solution concepts as the GNE or the Nash bargaining solution [21] should be shown.

Appendix

In this work, we use the notations in Table I. First, we recall the definitions of l.s.c. and u.s.c. (see p.109 in [4]).

G	original game $G = \big(\mathcal{N}, (\mathcal{A}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}}, (C_i)_{i\in\mathcal{N}}\big)$
$\mathcal{N}$	set of players
N	cardinality of set of players
$\mathcal{A}_i$	set of strategies of player $i \in \mathcal{N}$
$\mathcal{A}$	product $\prod_{i\in\mathcal{N}} \mathcal{A}_i$
$\mathcal{A}_{-i}$	product $\prod_{i\in\mathcal{N}, i\neq j} \mathcal{A}_j$
u_i	utility of player $i \in \mathcal{N}$
C_i	individual constraint correspondence of player $i \in \mathcal{N}$
$\mathcal{L}$	coupled constraint
L_i	coupled constraint correspondence of player $i \in \mathcal{N}$
G_ρ	ψ-penalized game $G_\rho = \big(\mathcal{N}, (\mathcal{A}_i)_{i\in\mathcal{N}}, (u_i - \rho\psi_i)_{i\in\mathcal{N}}, (L_i)_{i\in\mathcal{N}}\big)$
ρ	penalty parameter
ψ_i	penalty function of player $i \in \mathcal{N}$
G^{ce}	game serving as a counterexample in Section III-B
G^{DD}	game serving as an example in Section IV
$d(x, A)$	distance of a point to a set s.t. $d(x, A) := \inf_{a\in A} \lvert x - a\rvert$
$\mathcal{P}(\cdot)$	power set
$\Gamma(\cdot)$	graph of a function or correspondence
$\bar{A}$	closure of a set $A \subset \mathbb{R}$

TABLE I
NOTATIONS

First, we recall the definition of l.s.c. (see p.109 in [4]).

Definition 2: Let $C : X \to Y$ be a correspondence between topological spaces X and Y. Let x_0 be a point in X. C is said to be *lower semi-continuous* (l.s.c.) at x_0 if for each open set $U \subset Y$ with $C(x_0) \cap U \neq \emptyset$ there is a neighborhood $N(x_0) \subset X$ such that for all $x \in N(x_0)$ it holds $C(x) \cap U \neq \emptyset$. We say that C is l.s.c. in X if it is l.s.c. at x for all $x \in X$.

We remark that by Theorem 1 on p.109 in [4] and Corollary 1.1 in [19], the following equivalence holds.

Lemma 6: A correspondence $C : X \to Y$ between topological spaces X and Y is l.s.c. at $x_0 \in X$ if and only if for all sequences $(x_n)_{n\in\mathbb{N}}$ converging to x_0 and $y_0 \in C(x_0)$ there is $N \in \mathbb{N}$ and a sequence $(y_n)_{n\in\mathbb{N}}$ converging to y_0 such that $y_n \in C(x_n)$ for all $n \geq N$.

Now, we prove a preliminary result on l.s.c. correspondences.

Lemma 7: Let $C : X \to 2^Y$ be a correspondence between two topological spaces X and Y. If C is l.s.c. in x_0, the correspondence $\overline{C} : X \to Y$ defined by $\overline{C}(x) = \overline{C(x)}$ is also l.s.c. in x_0.

Proof: Let G be an open set such that $\overline{C}(x_0) \cap G \neq \emptyset$. Then there is some $x \in \overline{C}(x_0) \cap G$. Since $x \in G$ and G open, G is an open neighborhood of x. By the definition of the closure of a set, we have $C(x_0) \cap G \neq \emptyset$ (see p.5-6 in [24] and Theorem 3 in [4]). Furthermore, since C is l.s.c. in x_0, there is a neighborhood $U(x_0)$ such that for all $x \in U(x_0)$, it holds $C(x) \cap G \neq \emptyset$ and thus also $\overline{C}(x) \cap G \neq \emptyset$. We conclude that $\overline{C}$ is l.s.c. in x_0. ■

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